

NONEXISTENCE OF TWISTS AND SURGERIES GENERATING EXOTIC 4-MANIFOLDS

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ABSTRACT. It is well known that for any exotic pair of simply connected closed 4-manifolds, one is obtained by twisting the other along a contractible submanifold. In contrast, here we show that for each positive integer n , there exists an infinite family of pairwise exotic simply connected closed 4-manifolds such that, for any 4-manifold X and any compact (not necessarily connected) codimension zero submanifold W with $b_1(\partial W) < n$, the family cannot be generated by twisting X along W and varying the gluing map. As a corollary, we show that there exists no ‘universal’ 4-manifold with boundary such that any exotic family is generated by twisting along an embedded copy of the 4-manifold. Moreover, we give similar results for surgeries.

1. INTRODUCTION

A powerful method for constructing exotic (i.e. homeomorphic but not diffeomorphic) smooth structures on a compact 4-manifold is twisting the 4-manifold along a compact codimension zero submanifold, that is, removing the submanifold and regluing it differently. In fact, a well-known theorem states that for any exotic pair of simply connected closed smooth oriented 4-manifolds, one is obtained by twisting the other along a contractible submanifold via an involution on the boundary ([15], [30]), though there is still no known 4-manifold whose all exotic smooth structures are found. Such contractible 4-manifolds are called corks ([26], [7]), and they have various interesting applications (e.g. [1], [8], [10], [13], [6], [40]). Due to the order of the gluing map, they are often called order-2 corks ([34]).

Recently higher order corks ([35], [14]) and surprisingly infinite order corks ([23], see also [3], [24], [37]) were discovered, but interestingly Tange [36] showed that a natural extension of the above theorem on corks does not always hold for an infinite exotic family, by showing a certain finiteness for Ozsváth-Szabó invariants of cork-twisted 4-manifolds. More precisely, he gave infinite families of pairwise exotic closed 4-manifolds such that, for any 4-manifold X , any contractible submanifold W , and any self-diffeomorphism f of ∂W , the families cannot be produced by twisting X along W via powers of f .

In this paper we discuss generalizations of the existence theorem of corks from viewpoints of twists and more general surgeries.

1.1. Twists generating exotic families of 4-manifolds. We first discuss twists, using the following terminologies for convenience.

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Definition 1.1. A pair (X, W) consisting of a smooth oriented 4-manifold X and a compact (not necessarily connected) codimension zero submanifold W will be called a *4-manifold pair*. For a family of smooth oriented 4-manifolds, we say that *the family is generated by twisting X along W* , if each member is obtained from X by removing the submanifold W and gluing it back via an orientation-preserving self-diffeomorphism of ∂W .

For example, logarithmic transforms (i.e. twists along $T^2 \times D^2$) generate various interesting infinite exotic families (cf. [25], [19], [2]). Hence it would be natural to ask the following problem.

Problem 1.2. Given a family of homeomorphic but pairwise non-diffeomorphic compact connected oriented smooth 4-manifolds, does there exist a 4-manifold pair (X, W) such that the family is generated by twisting X along W ?

This problem generalizes the cork theorem, since we do not impose any restrictions on the topology of W and on the gluing map. We note that a self-diffeomorphism of ∂W may not preserve the connected components of ∂W when ∂W is disconnected, and hence twists in this problem are largely flexible when W is disconnected.

If the answer is affirmative, then we obtain a very useful approach for finding and understanding all exotic smooth structures on a 4-manifold, since they are generated by just a single submanifold in this case. However, we give a partial negative answer under a mild assumption on $b_1(\partial W)$.

Theorem 1.3. *For each positive integer n , there exists an infinite family of homeomorphic but pairwise non-diffeomorphic compact connected oriented smooth 4-manifolds such that, for any 4-manifold pair (X, W) with $b_1(\partial W) < n$, the family cannot be generated by twisting X along W . (Note that W and ∂W are not necessarily connected.) Furthermore, the family can be chosen so that each member is simply connected and closed.*

This result makes a sharp contrast with the aforementioned existence theorem of corks, since the boundary of any compact contractible 4-manifold is a (connected) homology 3-sphere and thus satisfies $b_1 = 0$. We note that the aforementioned Tange's result follows from this result.

To prove this theorem, we introduce the *adjunction genus*, which is a diffeomorphism invariant of 4-manifolds determined by the minimal genus function. We moreover give a sufficient condition that an infinite family is an example of this theorem, by using the adjunction genera. The sufficient condition gives various examples, and we here list a few simple ones in the $n = 1$ case.

Example 1.4. (1) *Elliptic surfaces.* Any infinite family of homeomorphic but pairwise non-diffeomorphic simply connected closed elliptic surfaces with $b_2^+ > 1$ is an example of the $n = 1$ case of Theorem 1.3.

(2) *Knot surgery.* Suppose that a simply connected closed oriented smooth 4-manifold X with $b_2^+ > 1$ contains a homologically non-trivial torus T of self-intersection 0 satisfying $\pi_1(X - T) \cong 1$ and has a non-trivial Seiberg-Witten invariant. Then for any infinite family $\{K_i\}$ of knots in S^3 whose Alexander polynomials have pairwise distinct degrees, the infinite family $\{X_{K_i}\}$ of homeomorphic but pairwise non-diffeomorphic simply connected closed 4-manifolds is an example of the $n = 1$ case of Theorem 1.3, where each X_{K_i} denotes the 4-manifold obtained from

X by Fintushel-Stern's knot surgery [18] along T using K_i .

(3) *Small Stein 4-manifolds.* The infinite families of homeomorphic but pairwise non-diffeomorphic simply connected compact Stein 4-manifolds with $b_2 = 2$ obtained in [11] and [39] are examples of the $n = 1$ case of Theorem 1.3.

One might hope the existence of a 'universal' 4-manifold W with boundary such that any exotic family is generated by twisting a 4-manifold along a copy of W , but the above theorem dashes this hope.

Corollary 1.5. *For any compact oriented smooth 4-manifold W with boundary, there exists an infinite family of homeomorphic but pairwise non-diffeomorphic simply connected closed oriented smooth 4-manifolds such that, for any smooth 4-manifold X and any embedded copy of W , the family cannot be generated by twisting X along the copy of W .*

1.2. Surgeries generating exotic families of 4-manifolds. Next we discuss surgeries, using the following terminology.

Definition 1.6. For a 4-manifold pair (X, W) and a family of smooth oriented 4-manifolds, we say that *the family is generated by performing surgeries on X along W* , if each member is obtained from X by removing the submanifold W and gluing a compact oriented smooth 4-manifold whose boundary is orientation-preserving diffeomorphic to ∂W . Note that we do not fix the newly glued piece.

Clearly surgeries can produce much more 4-manifolds than twists. For example, one can replace submanifolds with their exotic copies. Since surgeries produce various infinite exotic families (cf. [25], [19], [2]), it would be natural to consider a surgery version of Problem 1.2.

Problem 1.7. Given a family of homeomorphic but pairwise non-diffeomorphic compact connected oriented smooth 4-manifolds with $b_2 > 0$, do there exist a member X and a compact codimension zero submanifold W with $b_2(W) < b_2(X)$ such that the family is generated by performing surgeries on X along W ?

The condition $b_2(W) < b_2(X)$ might look unnatural, but we impose this restriction to avoid a trivial affirmative answer. Indeed, for any member X and any compact codimension zero submanifold V of the 4-ball, it is easy to see that the pair X and $W = X - \text{int } V$ provides an affirmative answer, realizing any integer not less than $b_2(X)$ as $b_2(W)$. We also note that choosing X from the members does not affect whether this problem is affirmative. This condition is added to avoid too much flexibility on the topology of X .

Here we show that this problem also has a partial negative answer under a mild assumption on $b_2(W) + 3b_1(\partial W)$, even though surgeries are much more flexible than twists.

Theorem 1.8. *For each positive integer n , there exists an infinite family of homeomorphic but pairwise non-diffeomorphic compact connected oriented smooth 4-manifolds with $b_2 > n$ such that, for any member X and any compact codimension zero submanifold W with $b_2(W) + 3b_1(\partial W) < n$, the family cannot be generated by performing surgeries on X along W . (Note that W and ∂W are not necessarily connected.) Furthermore, the family can be chosen so that each member is simply connected and closed.*

It would be worth to remark that the above family can be chosen so that it satisfies the condition of Theorem 1.3. We also give a sufficient condition that an infinite family satisfies the condition of this theorem by using adjunction genera, and we show that there are various examples. For example, all the families in Example 1.4 are examples of the $n = 1$ case of this theorem.

Similarly to the case of twists, Theorem 1.8 shows the nonexistence of a ‘universal’ 4-manifold with boundary even for surgeries.

Corollary 1.9. *For any compact oriented smooth 4-manifold W with boundary, there exists an infinite family of homeomorphic but pairwise non-diffeomorphic simply connected closed oriented smooth 4-manifolds such that, for any member X and any embedded copy of W , the family cannot be generated by performing surgeries on X along the copy of W .*

Finally we discuss a further generalization of the existence theorem of corks.

Problem 1.10. Given a family of homeomorphic but pairwise non-diffeomorphic compact connected oriented smooth 4-manifolds, do there exist compact oriented smooth 4-manifolds X and W such that the family is generated by twisting X along an embedded copy of W and varying the embedding of W ?

This problem is largely flexible than Problem 1.2, since we vary the embedding of W . A related problem was studied by Akbulut and the author in [8] and [9], and it was shown that many order-2 corks can produce infinite exotic families by twisting along the corks via the involutions and varying the embeddings, though they can produce only exotic pairs in the case where the embeddings are fixed. By contrast, we give a partial negative answer to this problem, by applying (the proof of) Theorem 1.8 to the complements of embedded copies of W .

Corollary 1.11. *For each positive integer n , there exists an infinite family of homeomorphic but pairwise non-diffeomorphic simply connected closed oriented smooth 4-manifolds with $b_2 = m > n$ such that, for any compact oriented smooth 4-manifolds X and W satisfying $m - n < b_2(W) - 4b_1(\partial W)$, the family cannot be generated by twisting X along an embedded copy of W and varying the embedding of W .*

To prove our main results, we give sufficient conditions that infinitely many members of a family of 4-manifolds have pairwise distinct adjunction genera, by applying the adjunction inequalities. As a corollary, we obtain a simple but effective method for ‘coarsely’ distinguishing smooth structures of 4-manifolds admitting Stein structures.

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2. ADJUNCTION GENUS

In this section, we introduce the adjunction genus and our conventions. We also briefly recall genus invariants of 4-manifolds.

For a smooth 4-manifold X and a second homology class α , the minimal genus $g_X(\alpha)$ of α is defined to be the minimal genus of a smoothly embedded closed oriented surface representing α (cf. [25]). Here note that the genus of a disconnected

closed oriented surface is defined to be the sum of the genera of all connected components. If X is connected, then for any disconnected closed oriented surface in X , one can construct a connected closed oriented surface of the same genus representing the same homology class. The function $g_X : H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ is called the *minimal genus function* of X . This function has useful information about smooth structures on 4-manifolds, but in general, it is significantly difficult to distinguish the functions of two 4-manifolds. The author [38] (cf. [39]) thus introduced the *relative genus function*, which is a genus invariant (i.e. a diffeomorphism invariant determined by the minimal genus function on a 4-manifold). Gompf [22] subsequently introduced a different genus invariant called *genus rank function*, and the author [41] recently introduced a new genus invariant called *intersection genus*. In this paper we introduce yet another genus invariant to prove our main results.

Here we introduce our conventions. Let X be an oriented smooth 4-manifold such that $H_2(X; \mathbb{Z})$ is finitely generated. If X is compact, then this condition clearly holds. A finite subset $\mathbf{v} = \{v_1, v_2, \dots, v_m\}$ of $H_2(X; \mathbb{Z})$ will be called a *rational basis* of $H_2(X; \mathbb{Z})$, if \mathbf{v} becomes a basis of $H_2(X; \mathbb{Q})$ through the natural homomorphism. In the case where $b_2(X) = 0$, we regard the empty set as the rational basis of $H_2(X; \mathbb{Z})$. We denote the self-intersection number of a second homology class v by $v \cdot v$.

Now we define the adjunction genus.

Definition 2.1. The function $g_X^{ad} : H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ defined by

$$g_X^{ad}(v) = \max\{2g_X(v) - v \cdot v, 0\}$$

will be called the *adjunction function* of X . For a rational basis $\mathbf{v} = \{v_1, v_2, \dots, v_m\}$ of $H_2(X; \mathbb{Z})$ and a positive integer $n \leq b_2(X)$, we define a non-negative integer $G_{X, \mathbf{v}, n}$ as the n -th largest value of $\{g_X^{ad}(v_1), g_X^{ad}(v_2), \dots, g_X^{ad}(v_m)\}$, where the order is counted with multiplicity. For a positive integer $n > b_2(X)$, we put $G_{X, \mathbf{v}, n} = 0$. For a non-negative integer n , we define an integer $G_X(n)$ by

$$G_X(n) = \begin{cases} \min\{G_{X, \mathbf{v}, n} \mid \mathbf{v} \text{ is a rational basis of } H_2(X; \mathbb{Z})\}, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0. \end{cases}$$

We call $G_X(n)$ the *adjunction n -genus* of X .

The adjunction n -genus is clearly a diffeomorphism invariant of oriented 4-manifolds for each fixed integer n . One can also define an integral version of the adjunction genus by using (ordinary) bases instead of rational bases, but we do not pursue this point here. In general, computing adjunction genera is a very hard problem, but estimating their values are useful for studying smooth properties of families of 4-manifolds.

Remark 2.2. For each $n \geq 2$, the inequality $G_X(n-1) \geq G_X(n)$ holds. Thus for any infinite family of 4-manifolds with pairwise distinct adjunction n -genera, at least infinitely many members have pairwise distinct adjunction $(n-1)$ -genera.

3. ADJUNCTION GENERA, TWISTS AND SURGERIES

In this section, utilizing adjunction genera, we give sufficient conditions that an infinite family of 4-manifolds cannot be generated by twists and/or surgeries. We begin with twists.

Theorem 3.1. *Suppose that infinitely many members of a family of compact connected oriented smooth 4-manifolds have pairwise distinct adjunction n -genera for a positive integer n . Then the family cannot be generated by twisting X along W for any 4-manifold pair (X, W) with $b_1(\partial W) < n$. (Note that W and ∂W are not necessarily connected.)*

Remark 3.2. By our definition, the gluing map $\partial W \rightarrow \partial W$ used in a twisting operation must preserve the orientation of ∂W , so that the orientation of W is preserved after the twist. However, one can use an orientation-reversing gluing map by gluing \overline{W} instead of W , where \overline{W} denotes the 4-manifold W equipped with the reverse orientation. (Precisely, this operation is a surgery.) As easily seen from the proof, this theorem still holds for this extended definition of twists.

To prove this theorem, we show the following finiteness.

Proposition 3.3. *Suppose that a family of compact connected oriented smooth 4-manifolds is generated by twisting X along W for a 4-manifold pair (X, W) . Then, for any integer $n > b_1(\partial W)$, there are at most finitely many integers that can be the values of the adjunction n -genera of the members.*

We first prove the $b_1(\partial W) = 0$ case, since its proof is short and nicely demonstrates our main idea.

Proof of the $b_1(\partial W) = 0$ case of Proposition 3.3. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family satisfying the assumption. We fix rational bases $\mathbf{v} = \{v_1, v_2, \dots, v_s\}$ and $\mathbf{w} = \{w_1, w_2, \dots, w_t\}$ of $H_2(X - W; \mathbb{Z})$ and $H_2(W; \mathbb{Z})$ respectively. We regard each v_i and w_j as elements of $H_2(X_\lambda; \mathbb{Z})$ via the inclusions $X - W \hookrightarrow X_\lambda$ and $W \hookrightarrow X_\lambda$. Due to the assumption $b_1(\partial W) = 0$, the Mayer-Vietoris exact sequence for each $X_\lambda = (X - W) \cup W$ implies that $\mathbf{v} \cup \mathbf{w}$ is a rational basis of $H_2(X_\lambda; \mathbb{Z})$. Since each v_i and w_j are represented by surfaces in $X - W$ and W , the definition of the adjunction 1-genus shows that

$$0 \leq G_{X_\lambda}(n) \leq G_{X_\lambda, \mathbf{v} \cup \mathbf{w}, n} \leq \max\{G_{X-W, \mathbf{v}, 1}, G_{W, \mathbf{w}, 1}\}.$$

Note that the value of $\max\{G_{X-W, \mathbf{v}, 1}, G_{W, \mathbf{w}, 1}\}$ is independent of λ . Hence at most finitely many integers can be the values of the adjunction 1-genera of the members. \square

Now we prove the general case.

Proof of Proposition 3.3. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family satisfying the assumption. We fix rational bases $\mathbf{v} = \{v_1, v_2, \dots, v_s\}$ and $\mathbf{w} = \{w_1, w_2, \dots, w_t\}$ of $H_2(X - W; \mathbb{Z})$ and $H_2(W; \mathbb{Z})$ respectively. We regard each v_i and w_j as elements of $H_2(X_\lambda; \mathbb{Z})$ via the inclusions $X - W \hookrightarrow X_\lambda$ and $W \hookrightarrow X_\lambda$. The Mayer-Vietoris exact sequence for $X_\lambda = (X - W) \cup W$ gives the following exact sequence.

$$H_2(W; \mathbb{Q}) \oplus H_2(X - W; \mathbb{Q}) \xrightarrow{\varphi_\lambda} H_2(X_\lambda; \mathbb{Q}) \xrightarrow{\partial_\lambda} H_1(\partial W; \mathbb{Q}).$$

This exact sequence shows that

$$b_2(X_\lambda) = \dim(\text{Im } \varphi_\lambda) + \dim(\text{Im } \partial_\lambda) \quad \text{and} \quad \dim(\text{Im } \partial_\lambda) \leq b_1(\partial W).$$

Now we estimate the value of $G_{X_\lambda}(n)$ for an integer $n > b_1(\partial W)$. If $\lambda \in \Lambda$ satisfies $\dim(\text{Im } \varphi_\lambda) = 0$, then we see that $b_2(X_\lambda) \leq b_1(\partial W) < n$. Hence we get $G_{X_\lambda}(n) = 0$ by the definition of the adjunction n -genus. Now suppose that

$\lambda \in \Lambda$ satisfies $\dim(\text{Im } \varphi_\lambda) = k$ for some $k > 0$. Then there exists a subset $\mathbf{u}_\lambda = \{u_1, u_2, \dots, u_k\}$ of $\mathbf{v} \cup \mathbf{w}$ such that u_1, u_2, \dots, u_k are linearly independent in $H_2(X_\lambda; \mathbb{Z})$, since \mathbf{v} and \mathbf{w} are rational bases of $H_2(X - W; \mathbb{Z})$ and $H_2(W; \mathbb{Z})$. Therefore there exists a (possibly empty) subset $\mathbf{x}_\lambda = \{x_1, x_2, \dots, x_l\}$ of $H_2(X_\lambda; \mathbb{Z})$ such that $\mathbf{u}_\lambda \cup \mathbf{x}_\lambda$ is a rational basis of $H_2(X_\lambda; \mathbb{Z})$. We here note that $l = \dim(\text{Im } \partial_\lambda) \leq b_1(\partial W) < n$. We thus see that

$$\begin{aligned} 0 \leq G_{X_\lambda}(n) &\leq G_{X_\lambda, \mathbf{u}_\lambda \cup \mathbf{x}_\lambda}(n) \\ &\leq \max\{g_{X_\lambda}(u_i) \mid 1 \leq i \leq k\} \leq \max\{G_{X-W, \mathbf{v}, 1}, G_{W, \mathbf{w}, 1}\}. \end{aligned}$$

Therefore at most finitely many integers can be the values of the adjunction n -genera of the members of $\{X_\lambda\}_{\lambda \in \Lambda}$. \square

Proof of Theorem 3.1. This is straightforward from Proposition 3.3. \square

We next discuss surgeries.

Theorem 3.4. *Suppose that infinitely many members of a family of compact connected oriented smooth 4-manifolds with $b_2 = m$ have pairwise distinct adjunction n -genera for a positive integer n . Then, for any 4-manifold pair (X, W) satisfying $m - b_2(X) + b_2(W) + 3b_1(\partial W) < n$, the family cannot be generated by performing surgeries on X along W . (Note that W and ∂W are not necessarily connected.)*

We show the following finiteness to prove this theorem.

Proposition 3.5. *Suppose that a family of compact connected oriented smooth 4-manifolds with $b_2 = m$ is generated by performing surgeries on X along W for a 4-manifold pair (X, W) . Then, for any positive integer $n > m - b_2(X - W) + 2b_1(\partial W)$, there are at most finitely many integers that can be the values of the adjunction n -genera of the members.*

We first prove the $b_1(\partial W) = 0$ case, since its proof is short and nicely demonstrates our main idea.

Proof of the $b_1(\partial W) = 0$ case of Proposition 3.5. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family satisfying the assumption. Then each X_λ has a decomposition $X_\lambda = (X - W) \cup W_\lambda$ for some compact oriented 4-manifold W_λ . We fix rational bases $\mathbf{v} = \{v_1, v_2, \dots, v_s\}$ and $\mathbf{w}_\lambda = \{w_1, w_2, \dots, w_t\}$ of $H_2(X - W; \mathbb{Z})$ and $H_2(W_\lambda; \mathbb{Z})$ respectively. By the assumption $b_1(\partial W) = 0$, we easily see that $\mathbf{v} \cup \mathbf{w}_\lambda$ is a rational basis of $H_2(X_\lambda; \mathbb{Z})$, where we regard \mathbf{v} and \mathbf{w}_λ as subsets of $H_2(X_\lambda; \mathbb{Z})$ via the inclusions $X - W \hookrightarrow X_\lambda$ and $W_\lambda \hookrightarrow X_\lambda$. It is thus easy to check

$$t = b_2(W_\lambda) = m - b_2(X - W) < n.$$

The definition of the adjunction n -genus together with this inequality implies that

$$0 \leq G_{X_\lambda}(n) \leq G_{X_\lambda, \mathbf{v} \cup \mathbf{w}_\lambda, n} \leq G_{X-W, \mathbf{v}, 1}.$$

Therefore at most finitely many integers can be the values of $G_{X_\lambda}(n)$'s. \square

Now we prove the general case.

Proof of Proposition 3.5. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family satisfying the assumption. Then each X_λ has a decomposition $X_\lambda = (X - W) \cup W_\lambda$ for some compact oriented 4-manifold W_λ . We fix rational bases $\mathbf{v} = \{v_1, v_2, \dots, v_s\}$ and $\mathbf{w}_\lambda = \{w_1, w_2, \dots, w_t\}$ of $H_2(X - W; \mathbb{Z})$ and $H_2(W_\lambda; \mathbb{Z})$ respectively. We regard each v_i and w_j as elements of $H_2(X_\lambda; \mathbb{Z})$ via the inclusions $X - W \hookrightarrow X_\lambda$ and $W_\lambda \hookrightarrow X_\lambda$.

We here estimate the value of $b_2(W_\lambda)$. The Mayer-Vietoris exact sequence for $X_\lambda = (X - W) \cup W_\lambda$ gives the following exact sequence.

$$H_2(\partial W; \mathbb{Q}) \xrightarrow{\varphi_\lambda} H_2(X - W; \mathbb{Q}) \oplus H_2(W_\lambda; \mathbb{Q}) \xrightarrow{\psi_\lambda} H_2(X_\lambda; \mathbb{Q}) \xrightarrow{\partial_\lambda} H_1(\partial W; \mathbb{Q}).$$

This exact sequence implies that

$$\begin{aligned} b_2(X_\lambda) &= \dim(\operatorname{Im} \psi_\lambda) + \dim(\operatorname{Im} \partial_\lambda), \quad \dim(\operatorname{Im} \partial_\lambda) \leq b_1(\partial W), \\ b_2(X - W) + b_2(W_\lambda) - b_1(\partial W) &\leq \dim(\operatorname{Im} \psi_\lambda). \end{aligned}$$

Hence we easily see

$$b_2(W_\lambda) \leq m - b_2(X - W) + b_1(\partial W).$$

Now we estimate the value of $G_{X_\lambda}(n)$ for a positive integer $n > m - b_2(X - W) + 2b_1(\partial W)$. If $\lambda \in \Lambda$ satisfies $\dim(\operatorname{Im} \psi_\lambda) < n - b_1(\partial W)$, then we easily see $b_2(X_\lambda) < n$. Hence we get $G_{X_\lambda}(n) = 0$ by the definition of the adjunction n -genus. Now suppose that $\lambda \in \Lambda$ satisfies $\dim(\operatorname{Im} \psi_\lambda) = k$ for some $k \geq n - b_1(\partial W)$. Then we see $k > m - b_2(X - W) + b_1(\partial W)$ due to the assumption on n . The condition $\dim(\operatorname{Im} \psi_\lambda) = k$ implies that there exists a subset $\mathbf{u}_\lambda = \{u_1, u_2, \dots, u_k\}$ of $\mathbf{v} \cup \mathbf{w}_\lambda$ such that u_1, u_2, \dots, u_k are linearly independent in $H_2(X_\lambda; \mathbb{Z})$, since \mathbf{v} and \mathbf{w}_λ are rational bases of $H_2(X - W; \mathbb{Z})$ and $H_2(W_\lambda; \mathbb{Z})$. Therefore there exists a (possibly empty) subset $\mathbf{x}_\lambda = \{x_1, x_2, \dots, x_l\}$ of $H_2(X_\lambda; \mathbb{Z})$ such that $\mathbf{u}_\lambda \cup \mathbf{x}_\lambda$ is a rational basis of $H_2(X_\lambda; \mathbb{Z})$. Note that $l = \dim(\operatorname{Im} \partial_\lambda) \leq b_1(\partial W)$. Due to the above estimate of $b_2(W_\lambda)$ and the assumption on n , we obtain

$$b_2(W_\lambda) + l < n.$$

This inequality shows that the number of elements of $(\mathbf{u}_\lambda \cap \mathbf{w}_\lambda) \cup \mathbf{x}_\lambda$ is less than n . We thus obtain the following estimate of $G_{X_\lambda}(n)$.

$$\begin{aligned} 0 \leq G_{X_\lambda}(n) &\leq G_{X_\lambda, \mathbf{u}_\lambda \cup \mathbf{x}_\lambda}(n) \\ &\leq \max\{g_{X_\lambda}(u_i) \mid u_i \in \mathbf{v}\} \leq G_{X-W, \mathbf{v}, 1}. \end{aligned}$$

Therefore, at most finitely many integers can be the values of the adjunction n -genera of the members of $\{X_\lambda\}_{\lambda \in \Lambda}$. \square

Proof of Theorem 3.4. For a 4-manifold pair (X, W) , we easily see that $b_2(X - W) \geq b_2(X) - b_2(W) - b_1(\partial W)$, similarly to the estimate of $b_2(W_\lambda)$ in the proof of Proposition 3.5. Hence if positive integers m, n satisfies $n > m - b_2(X) + b_2(W) + 3b_1(\partial W)$, then we see $n > m - b_2(X - W) + 2b_1(\partial W)$. Theorem 3.4 thus immediately follows from Proposition 3.5. \square

Corollary 3.6. *Suppose that infinitely many members of a family of compact connected oriented smooth 4-manifolds with $b_2 = m$ have pairwise distinct adjunction n -genera for a positive integer n . Then, for any compact oriented smooth 4-manifolds X and W with $m - n < b_2(W) - 4b_1(\partial W)$, the family cannot be generated by twisting X along an embedded copy of W and varying the embedding of W .*

Proof. Assume that an infinite family satisfies the assumption of this corollary. Suppose, to the contrary, that there exist compact oriented smooth 4-manifolds X and W with $m - n < b_2(W) - 4b_1(\partial W)$ such that the family is generated by twisting X along an embedded copy of W and varying the embedding of W . Let

W_0 be a copy of W embedded in X . Then the family is generated by performing surgeries on X along $X - \text{int } W_0$. We can easily check that

$$b_2(X - \text{int } W_0) \leq b_2(X) - b_2(W) + b_1(\partial W),$$

similarly to the estimate of $b_2(W_\lambda)$ in the proof of Proposition 3.5. Hence we see that

$$n > m - b_2(X) + b_2(X - \text{int } W_0) + 3b_1(\partial W).$$

Since infinitely many members of the family have pairwise distinct adjunction n -genera, this inequality contradicts Theorem 3.4. \square

Remark 3.7. As seen from the proofs, the results in this section still hold without the compactness condition of the members, if the second homology group of each member is finitely generated.

4. ESTIMATING ADJUNCTION GENERA

In this section, we give sufficient conditions that infinitely many members of a family of 4-manifolds to have pairwise distinct adjunction genera, by utilizing the adjunction inequalities. These conditions also provide simple but effective methods for coarsely distinguishing smooth structures.

Let $\{X_i\}_{i \in \mathbb{N}}$ be a family of compact connected oriented smooth 4-manifolds, and let S_i ($i \in \mathbb{N}$) be a finite subset of $H^2(X_i; \mathbb{Z})$. We introduce the following condition, which plays an important role for estimating the adjunction genera.

Definition 4.1. For a positive integer n , we say that the members of $\{S_i\}_{i \in \mathbb{N}}$ are pairwise n -nicely inequivalent, if there exist a positive integer $m_i \geq n$, non-zero classes $K_1^{(i)}, K_2^{(i)}, \dots, K_{m_i}^{(i)}$ of $H^2(X_i; \mathbb{Z})$, and non-negative integers $a_1^{(i)}, a_2^{(i)}, \dots, a_{m_i}^{(i)}$ for each $i \in \mathbb{N}$ such that the following conditions hold.

- $S_i = \{\pm a_1^{(i)} K_1^{(i)} \pm a_2^{(i)} K_2^{(i)} \pm \dots \pm a_{m_i}^{(i)} K_{m_i}^{(i)}\}$ for each $i \in \mathbb{N}$.
- $K_1^{(i)}, K_2^{(i)}, \dots, K_n^{(i)}$ are primitive and linearly independent for each $i \in \mathbb{N}$.
- For each $1 \leq j \leq n$, the integer sequence $\{a_j^{(i)}\}_{i \in \mathbb{N}}$ is strictly increasing.

We first discuss the case where the members are closed 4-manifolds. For a closed connected oriented smooth 4-manifold X with $b_2^+ > 1$, let β_X denote the set of Seiberg-Witten basic classes of X .

Theorem 4.2. *Let $\{X_i\}_{i \in \mathbb{N}}$ be an infinite family of connected closed oriented smooth 4-manifolds with $b_2^+ > 1$ which are of Seiberg-Witten simple types. Suppose each β_{X_i} has a subset S_i such that the members of $\{S_i\}_{i \in \mathbb{N}}$ are pairwise n -nicely inequivalent for a positive integer n . Then at least infinitely many members have pairwise distinct adjunction n -genera.*

Remark 4.3. Many families produced by logarithmic transforms or Fintushel-Stern's knot surgeries satisfy this assumption, as seen from the formulas of Seiberg-Witten invariants (cf. [19]).

Proof. By the assumption, there exist a positive integer $m_i \geq n$, non-zero classes $K_1^{(i)}, K_2^{(i)}, \dots, K_{m_i}^{(i)}$ of $H^2(X_i; \mathbb{Z})$, and non-negative integers $a_1^{(i)}, a_2^{(i)}, \dots, a_{m_i}^{(i)}$ for each $i \in \mathbb{N}$ such that the following conditions hold.

- (i) $S_i = \{\pm a_1^{(i)} K_1^{(i)} \pm a_2^{(i)} K_2^{(i)} \pm \dots \pm a_{m_i}^{(i)} K_{m_i}^{(i)}\}$ for each $i \in \mathbb{N}$.
- (ii) $K_1^{(i)}, K_2^{(i)}, \dots, K_n^{(i)}$ are primitive and linearly independent for each $i \in \mathbb{N}$.

(iii) For each $1 \leq j \leq n$, the integer sequence $\{a_j^{(i)}\}_{i \in \mathbb{N}}$ is strictly increasing.

Let $\mathbf{v}_i = \{v_1, \dots, v_k\}$ be a rational basis of $H_2(X_i; \mathbb{Z})$. Since $K_1^{(i)}, K_2^{(i)}, \dots, K_n^{(i)}$ are linearly independent, the universal coefficient theorem implies that there exist pairwise distinct elements $v_{t_1}, v_{t_2}, \dots, v_{t_n}$ of \mathbf{v}_i such that $\langle K_j^{(i)}, v_{t_j} \rangle \neq 0$ for each $1 \leq j \leq n$. We note that each element of S_i is a Seiberg-Witten basic class of X_i . Due to the condition (i), the adjunction inequalities ([27], [31], [32]) to v_{t_j} show that

$$\sum_{l=1}^{m_i} \left| \langle a_l^{(i)} K_l^{(i)}, v_{t_j} \rangle \right| + v_{t_j} \cdot v_{t_j} \leq \max\{0, 2g_{X_i}(v_{t_j}) - 2\},$$

implying $a_j^{(i)} \leq g_{X_i}^{ad}(v_{t_j})$ for each $1 \leq j \leq n$. It thus follows that

$$\min\{a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)}\} \leq \min\{g_{X_i}^{ad}(v_{t_1}), g_{X_i}^{ad}(v_{t_2}), \dots, g_{X_i}^{ad}(v_{t_n})\} \leq G_{X_i, \mathbf{v}, n}.$$

Hence we obtain

$$\min\{a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)}\} \leq G_{X_i}(n).$$

The condition (iii) thus shows $\lim_{i \rightarrow \infty} G_{X_i}(n) = \infty$. Therefore at least infinitely many members of $\{X_i\}_{i \in \mathbb{N}}$ have pairwise distinct adjunction n -genera. \square

We next discuss the case where the members admit Stein structures. For a compact connected oriented smooth 4-manifold X with boundary, we define a set $\mathcal{C}_S(X)$ by

$$\mathcal{C}_S(X) = \{c_1(X, J) \in H^2(X; \mathbb{Z}) \mid J \text{ is a Stein structure on } X\},$$

where $c_1(X, J)$ denotes the first Chern class of X equipped with a Stein structure J .

Theorem 4.4. *Let $\{X_i\}_{i \in \mathbb{N}}$ be an infinite family of compact connected oriented smooth 4-manifolds with boundary. Suppose each $\mathcal{C}_S(X_i)$ has a subset S_i such that the members of $\{S_i\}_{i \in \mathbb{N}}$ are pairwise n -nicely inequivalent for a positive integer n . Then at least infinitely many members have pairwise distinct adjunction n -genera.*

Proof. Since the adjunction inequality holds for any compact Stein 4-manifold ([28], [4], [29]), the proof of Theorem 4.2 works in this case as well. \square

This sufficient condition is of independent interest, since it is a simple sufficient condition that an infinite family has infinitely many pairwise non-diffeomorphic members. Note that the adjunction genus is a diffeomorphism invariant.

We here recall that the *divisibility* $d(\alpha)$ of an element α in a finitely generated abelian group G is defined by

$$d(\alpha) = \begin{cases} \max\{n \in \mathbb{Z} \mid \alpha = n\alpha' \text{ for some } \alpha' \in G\}, & \text{if } \alpha \text{ is not torsion;} \\ 0, & \text{if } \alpha \text{ is torsion.} \end{cases}$$

We immediately obtain the following corollary.

Corollary 4.5. *For a given infinite family of compact Stein 4-manifolds, if the divisibilities of the first Chern classes of the members are pairwise distinct, then at least infinitely many members have pairwise distinct adjunction 1-genera.*

5. EXAMPLES

In this section, we give simple examples of our main results. More precisely, we prove the following theorem using results in Section 4.

Theorem 5.1. *For each positive integer n , there exists an infinite family of homeomorphic but pairwise non-diffeomorphic compact connected oriented smooth 4-manifolds with pairwise distinct adjunction n -genera. Furthermore, the family can be chosen so that each member is simply connected and closed.*

By results in Section 3, we immediately see that the above examples give our main results.

Proof of Theorem 1.3. This follows from Theorems 5.1 and 3.1. □

Proof of Theorem 1.8. This follows from Theorems 5.1 and 3.4. □

Proof of Corollary 1.11. This follows from Theorem 5.1 and Corollary 3.6. □

In the rest we prove Theorem 5.1. Here we introduce our notations. Let $E(n)$ be the simply connected elliptic surface with Euler characteristic $12n$ and with no multiple fibers, and let $E(n)_{p,q}$ be the elliptic surface obtained from $E(n)$ by performing logarithmic transformations of multiplicities p and q along two regular fibers. We denote the Gompf nucleus [20] of $E(n)$ by $N(n)$, namely, $N(n)$ is the regular neighborhood of the union of a cusp fiber and a $-n$ -section of $E(n)$.

5.1. Examples of the $n = 1$ case of Theorem 5.1. Note that the following examples give a proof of Example 1.4.

5.1.1. Elliptic surfaces. Any infinite family of homeomorphic but pairwise non-diffeomorphic simply connected closed elliptic surfaces with $b_2^+ > 1$ has an infinite subfamily that is an example of the $n = 1$ case of Theorem 5.1. This can be seen as follows. As is well-known (cf. [25]), any simply connected elliptic surface with $b_2^+ > 1$ is diffeomorphic to $E(k)_{p,q}$ for some positive integers $k \geq 2$ and $p, q \geq 1$. Since the members of the infinite family are pairwise homeomorphic, the value of k is independent of a member. According to the computation of $SW_{E(k)_{p,q}}$ in [17] (see Theorem 3.3.6 in [25]), each $E(k)_{p,q}$ has a Seiberg-Witten basic class whose divisibility is $kpq - p - q$. Theorem 4.2 thus shows that infinitely many members have pairwise distinct adjunction 1-genera.

5.1.2. Knot surgery. Suppose that a simply connected closed oriented smooth 4-manifold X with $b_2^+ > 1$ contains a homologically non-trivial torus T of self-intersection 0 satisfying $\pi_1(X - T) \cong 1$ and has a non-trivial Seiberg-Witten invariant (e.g. elliptic surfaces). Let X_K denote the 4-manifold obtained from X by performing a Fintushel-Stern's knot surgery [18] along T using a knot K in S^3 . Then for any infinite family $\{K_i\}_{i \in \mathbb{N}}$ of knots in S^3 whose Alexander polynomials have pairwise distinct degrees, the members of the infinite family $\{X_{K_i}\}_{i \in \mathbb{N}}$ are homeomorphic but pairwise non-diffeomorphic simply connected closed 4-manifolds due to [18] and [33]. Furthermore, using the Fintushel-Stern's knot surgery formula ([18], cf. [16]), one can see that each $\beta_{X_{K_i}}$ has a subset S_i such that the members of $\{S_i\}_{i \in \mathbb{N}}$ are pairwise 1-nicely inequivalent. Therefore by Theorem 4.2, an infinite subfamily of $\{X_{K_i}\}_{i \in \mathbb{N}}$ is an example of the $n = 1$ case of Theorem 5.1.

5.1.3. Small Stein 4-manifolds. The infinite families of homeomorphic but pairwise non-diffeomorphic simply connected compact Stein 4-manifolds with $b_2 = 2$ obtained in [11] and [39] have infinite subfamilies that are examples of the $n = 1$ case of Theorem 1.3. Here we show this using different Stein structures obtained in [12] (The original Stein structures obtained in [11] and [39] works as well, but those in [12] are the simplest regarding our purpose). Let $m = (m_0, m_1, m_2)$ be a 3-tuple of integers satisfying $m_0 \geq 2$, $m_1 \geq 2$, $m_2 \geq 1$, and let $X^{(m)}$ and $X_p^{(m)}$ be the 4-manifolds in Figure 1 of [12], where p is a positive integer. Then, for each fixed m , infinitely many members of $\{X_{2p}^{(m)}\}_{p \in \mathbb{N}}$ are homeomorphic but pairwise non-diffeomorphic Stein 4-manifolds with $b_2 = 2$ ([11], [39]), where the Stein structure of each $X_p^{(m)}$ is the one given by Figure 5 of [12]. By the formula of the first Chern classes of Stein 4-manifolds in [21] together with Lemma 4.1 in [12], we can easily check that the divisibility of $c_1(X_p^{(m)})$ is $p(m_1 - 1) + m_0 - 2$. Hence Corollary 4.5 shows that an infinite subfamily of $\{X_{2p}^{(m)}\}_{p \in \mathbb{N}}$ is an example of the $n = 1$ case of Theorem 5.1.

5.2. Examples of the general case of Theorem 5.1. In the rest, we fix a positive integer n .

5.2.1. Knot surgery. Suppose that X is a simply connected closed oriented smooth 4-manifold with $b_2^+ > 1$ that has a non-trivial Seiberg-Witten invariant and contains pairwise disjoint n copies of the Gompf nucleus $N(2)$. For example, the elliptic surface $E(k)$ ($k \geq 2$) contains pairwise disjoint $2(k - 1)$ copies of the Gompf nucleus $N(2)$ (see Exercises 3.1.12.(c) in [25]). One can also construct many examples of X , since the boundary connected sum $\natural_n N(2)$ admits a Stein structure, and any compact Stein 4-manifold can be embedded into a simply connected closed minimal symplectic 4-manifold with $b_2^+ > 1$ ([5]).

For each $1 \leq i \leq n$, let T_i be a regular fiber of a cusp neighborhood of the i -th copy of $N(2)$ embedded in X . Then each T_i is a torus with the self-intersection number 0, and T_1, T_2, \dots, T_n represent linearly independent second homology classes of X . For an infinite family $\{K_i\}_{i \in \mathbb{N}}$ of knots in S^3 whose Alexander polynomials have pairwise distinct degrees, let X_i be the 4-manifold obtained by performing knot surgeries on X along the tori T_1, \dots, T_n using the knot K_i . Then each X_i is homeomorphic to X . On the other hand, due to the knot surgery formula ([18]), one can see that β_{X_i} has a subset S_i such that the members of $\{S_i\}_{i \in \mathbb{N}}$ are pairwise n -nicely inequivalent. Therefore Theorem 4.2 implies that an infinite subfamily of $\{X_i\}_{i \in \mathbb{N}}$ is an example of Theorem 5.1.

5.2.2. Stein 4-manifolds. We use the notation in Section 5.1.3, and we fix m . We here observe that $X_p^{(m)}$ admits (at least) two Stein structures: one is given by Figure 5 of [12], and the other is obtained from the one in Figure 5 by moving the $(-m_1 + 1)$ -box in the left side of the figure to the right side in a symmetric way (see also Figure 3). We easily see that the first Chern classes of these two Stein structures are $r(p, m)K^p$ and $-r(p, m)K^p$ for some primitive second cohomology class K^p of $X_p^{(m)}$, where $r(p, m) := (p(m_1 - 1) + m_0 - 2)$. Now let $Z_{n, 2p}$ be the boundary connected sum $\natural_n X_{2p}^{(m)}$. Then the members of $\{Z_{n, 2p}\}_{p \in \mathbb{N}}$ are pairwise homeomorphic. By the above observation, we see that $\mathcal{C}_S(X_p^{(m)})$ has the subset $S_p := \{r(p, m)(\pm K_1^p \pm K_2^p \pm \dots \pm K_n^p)\}$, where each K_i^p is the class of $H^2(Z_{n, 2p}; \mathbb{Z}) \cong$

$\oplus_n H^2(X_{2p}^{(m)}; \mathbb{Z})$ corresponding to K^p in the i -th copy of $H^2(X_{2p}^{(m)}; \mathbb{Z})$. Hence the members of $\{S_p\}_{p \in \mathbb{N}}$ are pairwise n -nicely inequivalent. Thus by Theorem 4.4, we see that an infinite subfamily of $\{Z_{n,2p}\}_{p \in \mathbb{N}}$ is an example of Theorem 5.1.

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